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NUMERICAL SOLUTION OF A CLASS OF BOUNDARY VALUE PROBLEMS ARISING IN THE PHYSICS OF JOSEPHSON JUNCTIONS

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Мелемов Х. Т., Бояджиев Т. Л. E11-2008-143
Численное решение класса краевых задач,
возникающих в физике джозефсоновских контактов

В настоящей работе предложен метод численного решения краевых задач для систем нелинейных дифференциальных уравнений, заданных на вложенных интервалах изменения независимой переменной. Алгоритм основывается на непрерывном аналоге метода Ньютона. Численное решение соответствующих линейных краевых задач на каждой итерации проводится методом сплайн-коллокации.

В качестве конкретного примера рассматривается задача о возможных распределениях магнитного потока в двухслойном джозефсоновском контакте, отдельные субконтакты которого имеют разные длины. Рассмотрено влияние отношения длин субконтактов на физические характеристики некоторых основных пар распределений в системе.

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in the Physics of Josephson Junctions

In this paper we propose a method of numerical solution of non-linear boundary value problems for systems of ODEs given on the embedded intervals. The algorithm is based on the continuous analog of Newton method coupled with spline-collocation scheme of fourth order of accuracy.

Demonstrative examples of similar problems take place in physics of stacked Josephson junctions with different layers lengths. As a concrete example we consider the problem of calculating the possible distributions of magnetic flux in a system of two magnetically coupled long Josephson junctions. The influence of length ratio on the main physical properties of basic bound states is investigated numerically. The existence of bifurcations by change of the lengths of the layers for some couples of solutions has been proved.

The investigation has been performed at the Laboratory of Information Technologies, JINR.

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1. PROBLEM STATEMENT

Non-linear systems of ordinary differential equations given on embedded intervals occur in many physical problems and especially in the theory of stacked Josephson junctions (JJ) [1–4].

We consider a model of two-layer JJ [5] with different layer lengths $2L$ and $2l$ correspondingly, where $L \geqslant l$. We suppose that the short layer is situated symmetrically to the long one. The coordinate origin is in the middle of the stack; i.e., for the long (first) layer we have $x \in [-L, L]$ and for the short (second) one $x \in [-l, l]$. We denote by $s \in (-1, 0]$ the coupling coefficient between the layers [1].

In order to obtain the model equations for stacked JJ, we consider the full energy functional $F[\varphi]$ which can be represented as a sum

$$F = F_1[\varphi_1] + F_2[\varphi_2] + F_{12}[\varphi]. \tag{1.1}$$

Here $\varphi(x) = [\varphi_1(x), \varphi_2(x)]^T$ is the vector of magnetic fluxes in the layers (the superscript T means transposition), $F_i[\varphi_i]$, $i = 1, 2$, are the partial energies of the uncoupled layers ($s = 0$). The functional $F_{12}[\varphi]$ represents the coupling energy between the layers. In the symmetric overlap case [5] the corresponding expressions can be represented in the form

$$F_1[\varphi_1] = \int\limits_{-L}^{L} \left(\frac{1}{2}\,\varphi_{1,x}^2 + 1 - \cos\varphi_1 - \gamma\varphi_1 \right) dx - h_e \Delta\varphi_1, \tag{1.2a}$$

$$F_2[\varphi_2] = \int\limits_{-l}^{l} \left[\frac{1}{2}\,\varphi_{2,x}^2 + \rho\,(1 - \cos\varphi_2) - \gamma\varphi_2 \right] dx - h_e \Delta\varphi_2, \tag{1.2b}$$

$$F_{12}[\varphi_1, \varphi_2] = \frac{s}{1 - s^2} \int\limits_{-l}^{l} \left[\frac{s}{2}\left(\varphi_{1,x}^2 + \varphi_{2,x}^2\right) - \varphi_{1,x}\varphi_{2,x} \right] dx + $$

$$+ \frac{s}{1+s}\, h_e \left[\varphi_1(l) - \varphi_1(-l) + \Delta\varphi_2 \right], \tag{1.2c}$$

where h_e — the external magnetic field, γ — the external current and vector $\Gamma = \gamma\,(1,1)^T$. The full magnetic fluxes across the layers are defined by

$$\Delta\varphi_1 = \varphi_1(L) - \varphi_1(-L), \quad \Delta\varphi_2 = \varphi_2(l) - \varphi_2(-l). \tag{1.3}$$

From the necessary extremum conditions [12] of the functional (1.1) we obtain the following non-linear boundary value problem (BVP):

$$\varphi_{1,x}(-L) = h_e\,, \tag{1.4a}$$

$$-\varphi_{1,xx} + \sin\varphi_1 - \gamma = 0, \quad x \in (-L,-l)\,, \tag{1.4b}$$

$$\varphi_{2,x}(-l+0) - s\varphi_{1,x}(-l+0) = (1-s)h_e\,, \tag{1.4c}$$

$$-A\varphi_{xx} + J_z(\varphi) + \Gamma = 0\,, \quad x \in (-l,l)\,, \tag{1.4d}$$

$$\varphi_{2,x}(l-0) - s\varphi_{1,x}(l-0) = (1-s)h_e\,, \tag{1.4e}$$

$$-\varphi_{1,xx} + \sin\varphi_1 - \gamma = 0\,, \quad x \in (l,L)\,, \tag{1.4f}$$

$$\varphi_{1,x}(L) = h_e\,. \tag{1.4g}$$

The square interaction matrix A depends only on coupling coefficient [3]:

$$A(s) = \frac{1}{1-s^2}\begin{pmatrix} 1 & -s \\ -s & 1 \end{pmatrix}.$$

We denote by $J_z = (\sin\varphi_1, \rho\sin\varphi_2)^T$ the Josephson currents vector. Physically, the parameter $\rho = L/l \geqslant 1$ represents the amplitude of Josephson current in the short layer. All the quantities are in dimensionless form (see, for example, [6]).

Equations (1.4a) and (1.4g) are the corresponding boundary conditions for $\varphi_1(x)$ at the boundaries $x = \pm L$, and (1.4c) and (1.4e) are the boundary conditions for $\varphi_2(x)$ at $x = \pm l$. On the boundaries $x = \pm l$ the standard smoothing conditions for $\varphi_1(x)$ are fulfilled.

When $L = l$ ($\rho = 1$) the traditional BVP [7] for two-layer JJ follows from (1.4).

2. SOLUTION ALGORITHM

The solution of non-linear BVP (1.4) is based on the continuous analog of Newton method [8]. At each iteration we solve the following linear BVP:

$$w_{1,x}(-L) = -\varphi_{1,x}(-L) + h_e, \tag{2.1a}$$

$$-w_{1,xx} + \cos\varphi_1 w_1 = \varphi_{1,xx} - \sin\varphi_1 + \gamma, \tag{2.1b}$$

$$w_{2,x}\,(-l+0) - s\,w_{1,x}\,(-l+0) =$$

$$= -\varphi_{2,x}\,(-l+0) + s\varphi_{1,x}\,(-l-0) + (1-s)\,h_e, \tag{2.1c}$$

$$-A(s)w_{xx} + Q(x)w = A(s)\varphi_{xx} - J_z(\varphi) - \Gamma, \tag{2.1d}$$

$$w_{2,x}(l-0) - s\,w_{1,x}(l-0) =$$

$$= -\varphi_{2,x}(l-0) + s\varphi_{1,x}(l-0) + (1-s)\,h_e, \tag{2.1e}$$

$$-w_{1,xx} + \cos\varphi_1 w_1 = \varphi_{1,xx} - \sin\varphi_1 + \gamma, \tag{2.1f}$$

$$w_{1,x}(L) = -\varphi_{1,x}(L) + h_e, \tag{2.1g}$$

where 2-matrix $Q(x)$ is defined by $Q(x) = \mathrm{diag}(\cos\varphi_1(x), \rho\cos\varphi_2(x))$.

For numerical solution of the problem (2.1) formally we can extend smoothly the «short» function $\varphi_2(x)$ on the interval $[-L, L]$ and then apply usual discretization technique to this «extended» BVP. But in this way there will be unduly null in the matrix of the algebraic linear system which increases its dimension.

In order to solve the linear problem (2.1), in this paper the spline-collocation scheme [9] is applied.

Let in interval $[-L, L]$ an irregular grid be given

$$\{x_i, i = 1, 2, \ldots, n, x_{i+1} = x_i + h_i, x_1 = -L, x_k = -l, x_r = l, x_n = L\}$$

with n nodes and steps h_i, $1 \leqslant k \leqslant r \leqslant n$. Note that boundary points $x = \pm l$ are included as nodes in the grid. The case $k = 1$ and $r = n$ corresponds to usual JJ with equal layers.

In every subinterval $[x_i, x_{i+1}]$, the solution is searched as a cube Hermitian spline [10]

$$S_1(x) = \Phi(t)\,u_i + \Psi(t)\,h_i\,m_i + \bar\Phi(t)\,u_{i+1} + \bar\Psi(t)\,h_i\,m_{i+1}, \tag{2.2}$$

where $t = (x - x_i)/h_i$, $t \in [0, 1]$ is local coordinate, $\{u_i, m_i\}$ — value of the spline $S_1(x)$ and their derivative $m(x) \equiv S_{1,x}(x)$ in nodes $i = 1, 2, \ldots, k-1$ and $i = r+1, \ldots, n$ of the grid. Basic functions $\Phi(t) = (1-t)^2(1+2t)$ and $\Psi(t) = t\,(1-t)^2$ satisfying conditions $\Phi(0) = 1$ and $\dot\Psi(0) = 1$ (super-dot indicates differentiation with respect to local variable t). Remaining values of basic functions and their derivatives in nodes are equal to zero. About functions $\bar\Phi(t)$ and $\bar\Psi(t)$ we have $\bar\Phi(t) = \Phi(1-t)$, $\bar\Psi(t) = -\Psi(1-t)$.

Similarly, for subintervals in $[-l, l]$ an approximate solution is searched in the form

$$S(x) = \begin{pmatrix} S_1(x) \\ S_2(x) \end{pmatrix} = \Phi(t) \begin{pmatrix} u_{1,i} \\ u_{2,i} \end{pmatrix} + \Psi(t) h_i \begin{pmatrix} m_{1,i} \\ m_{2,i} \end{pmatrix} +$$

$$+\bar{\Phi}(t)\begin{pmatrix} u_{1,i+1} \\ u_{2,i+1} \end{pmatrix} + \bar{\Psi}(t)h_i\begin{pmatrix} m_{1,i+1} \\ m_{2,i+1} \end{pmatrix}.$$

We choose the collocation dots to be Gaussian nodes $t_j = (1 \pm \sqrt{3}/3)/2$, $j = 1, 2$ in $[0, 1]$. Simultaneously with smoothing conditions for the unknown functions and accounting all the boundary conditions, we obtain the following block-diagonal system of algebraic equations:

$$WU = P.$$

Here U — vector of nodes variables and matrix W has a structure

$$W = \begin{pmatrix} e_0 & 0 & 0 & . & . & . & . & . & . & . \\ A_1 & B_1 & 0 & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & A_{k-1} & B_{k-1} & 0 & 0 & . & . & . & . \\ . & . & 0 & e_s & e_0 & 0 & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & A_s & B_s & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & e_s & e_0 & 0 & 0 & . \\ . & . & . & . & . & A_r & 0 & B_r & 0 & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & 0 & 0 & A_{n-1} & B_{n-1} \\ . & . & . & . & . & . & 0 & 0 & 0 & e_0 \end{pmatrix},$$

where 2-vectors $e_0 = (0, 1)$, $e_s = (0, -s) = -se_0$, and remaining elements of W are 2-matrices with elements

$$A_{i,j1} = -\frac{1}{h_i^2}\ddot{\Phi}_j + \Phi_j c_{ij}, \quad A_{i,j2} = -\frac{1}{h_i}\ddot{\Psi}_j + \Psi_j h_i c_{ij},$$
$$B_{i,j1} = -\frac{1}{h_i^2}\ddot{\bar{\Phi}}_j + \bar{\Phi}_j c_{ij}, \quad B_{i,j2} = -\frac{1}{h_i}\ddot{\bar{\Psi}}_j a_{ij} + \bar{\Psi}_j h_i c_{ij},$$

for $i = 1, \ldots, k-1$ and $i = r, \ldots, n-1$, and $c_{ij} = \cos\varphi_1(x_{ij})$. Elements W in $i = k, \ldots, r-1$, $j = 1, 2$ are 4-matrices with elements

$$[A_{i,j1}]_{mn} = -\frac{1}{h_i^2}\ddot{\Phi}_j a_{mn} + \Phi_j q_{mn,ij}, \quad [A_{i,j2}]_{mn} = -\frac{1}{h_i}\ddot{\Psi}_j a_{mn} + \Psi_j h_i q_{mn,ij},$$
$$[B_{i,j1}]_{mn} = -\frac{1}{h_i^2}\ddot{\bar{\Phi}}_j a_{mn} + \bar{\Phi}_j q_{mn,ij}, \quad [B_{i,j2}]_{mn} = -\frac{1}{h_i}\ddot{\bar{\Psi}}_j a_{mn} + \bar{\Psi}_j h_i q_{mn,ij}.$$

Values $\{a_{mn}\}$, $m, n = 1, 2$ are elements of matrix A and $\{q_{mn,ij}\}$ are elements of $Q(x)$ in the relating Gauss nodes.

The number of the blocks of matrix W is equal to the number $n-1$ of grid's subintervals. The number of the columns in every block is fixed (8 in the case under consideration), the number of rows in blocks depends on the number of the blocks. In the external intervals $x \in [-L, -l]$ and $x \in [l, L]$ every block has two rows except the first and last blocks. These two blocks contain additional rows, which take into account the boundary conditions (1.4a) and (1.4g) in long subjunction. The internal blocks in the intervals $x \in [-l, l]$ have four rows. The blocks corresponding to boundaries $x = \pm l$ have three rows: two rows from discretization (1.4b) and (1.4f), and one row from boundary conditions (1.4c) and (1.4e) for function $\varphi_2(x)$. The number of the node's variables is $2(k-1) + 4(r-k+1) + 2(n-r)$.

In order to solve this system of algebraic equations, we a use specialized subprogram CWIDTH, which is described in detail in [10]. This program realizes Gauss method modified for block-diagonal systems of algebraic equations.

Based on the represented algorithm, a program for investigation of static distributions of magnetic flux in two-layer JJs with different layer lengths is made.

3. NUMERICAL RESULTS

Further we will discuss some numerical results obtained by means of the algorithm discussed above.

The solutions of the boundary problem (1.4) depend on the coordinate x, as well as on the set of parameters $p \equiv (L, l, s, h_e, \gamma)$, i.e. $\varphi_i = \varphi_i(x, p)$, $i = 1, 2$. Further the dependence on p is denoted only if it is necessary.

The basic numerical characteristics of every solution of non-linear BVP (1.4) are full (1.1), partial (1.2a), (1.2b) and coupling energies (1.2c), the full magnetic fluxes through the layers (1.3), as well as the average magnetic fluxes [8]:

$$\begin{aligned} N_1(p) = N[\varphi_1] &= \frac{1}{2L\pi} \int_{-L}^{L} \varphi_1(x, p)\, dx, \\ N_2(p) = N[\varphi_2] &= \frac{1}{2l\pi} \int_{-l}^{l} \varphi_2(x, p)\, dx. \end{aligned} \tag{3.1}$$

The calculation of the solutions of BVP (1.4) and their possible bifurcations at change of the parameters is an important but difficult problem. In this paper we investigate the influence of the length of short contact $2l$ on some typical bound states in the stack. All numerical results are obtained for «long enough» contact ($2L = 10$) and fixed coupling coefficient $s = -0.3$.

The fluxon (vortex) distributions of magnetic flux play an important role in the theory and application of JJs. It is well known that in the «infinity» JJ, which is described by unperturbed sine-Gordon equation, there exists a countable set of solutions [11].

For physical reasons it is convenient to discriminate unipolar fluxon solutions, composed of equally oriented vortices of magnetic field and heteropolar solutions, whose internal magnetic field is a result of non-linear interaction between heteropolar vortices. Further we shall consider only simple unipolar solutions of kind $\Phi^{\pm n}$, where $n = 1, 2, \dots$

In case of finite length JJs, the possible solutions become deformed as a result of interactions with the boundaries, as well as with the applied external magnetic field h_e and external current γ [8]. But thus the values of the average magnetic fields remain constants

$$N\left[\Phi^{\pm n}\right] = N\left[\Phi_{\infty}^{\pm n}\right] = \pm n.$$

Here $\Phi_{\infty}^{\pm n}$ refers to corresponding solutions in the «infinity» contact.

A special case in two-layer JJs are vortex solutions of type $(\Phi^{\pm 1},\ \Phi^{\pm 1})$, composed of two unipolar fluxons, on one in each layer. Concrete examples are presented in Fig. 1 where $2L = 10$, $2l = 8$, $h_e = 0$ and $\gamma = 0$ (solid and dashed lines correspond to the states in the long and short layers). It is obvious that the length decreasing is compensated by rise of the amplitude of internal magnetic field $\varphi_{2,x}(x)$ in the short layer.

The influence of half-length l of the short layer on internal magnetic field for solutions of type (Φ^1, Φ^1) is presented in Figs. 1 and 2. The decrease of l increases the amplitude of the internal magnetic field $\varphi_{2,x}(x)$ in the short layer. In turn, the interaction between the layers leads to $\varphi_{2,x}(\pm l) \neq 0$, so the smoothness of the field $\varphi_{1,x}(x)$ in points $\pm l$ worsens and the inductive current $\varphi_{1,xx}(\pm l)$

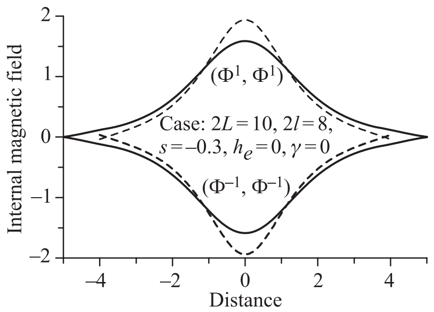

Fig. 1. $(\Phi^{\pm 1}, \Phi^{\pm 1})$-distributions of the internal magnetic field

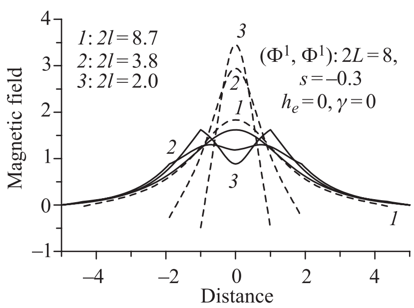

Fig. 2. (Φ^1, Φ^1)-states for $h_e = 0$, $\gamma = 0$, and different l

becomes broken. According to (1.4c) and (1.4e), it follows that the corresponding jump depends mainly on coupling coefficient s. This effect increases when the parameter l decreases (see Fig. 2). For large enough values of l the graph of $\varphi_{1,x}(x)$ has an extremum in the middle of contact (see curve *1*). The amplitude of this extremum decreases when half-length l decreases as well. For $2l \sim 5 \div 6$ the graph of the magnetic field $\varphi_{1,x}(x)$ has a plateau (curve *2*). At further decrease of l the graph of $\varphi_{1,x}(x)$ gets a minimum in the middle of JJ (curve *3*).

The feature specified above well explains the behaviour of curves $F_{12}(l)$, which are shown in Figs. 3 and 4. Really, for (Φ^1, Φ^1)-state and in general, for every unipolar couple of vortices, the integrand in (1.2c) is always negative. But because of a negative factor $s/(1-s^2)$, the integral remains always positive (see the dashed line in Fig. 3). Note that when $2l < 1.53$ for $h_e = 0$ and $\gamma = 0$ the couple (Φ^1, Φ^1) does not exist. Hence, the value $2l_B \approx 1.53$ is a bifurcation point for this solution at change of l. Physically, this bifurcation means that l_B is the minimal length providing existence of the solution (Φ^1, Φ^1). In case of single JJ, this fact is noticed in [13, 14].

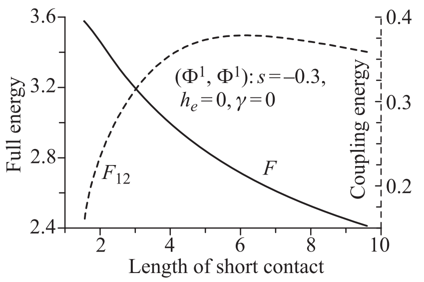

Fig. 3. Full energy of (Φ^1, Φ^1)-state

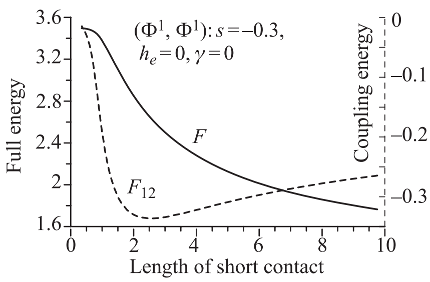

Fig. 4. Full energy of (Φ^1, Φ^{-1})-state

For heteropolar states the integrand in (1.2c) can vary — the first term is negative, but the second one is positive. Especially for $\left(\Phi^1, \Phi^{-1}\right)$ the integral remains always negative as at decrease of l (dashed line in Fig. 4).

A comparison of the dependences of partial energies $F_i(l)$, $i = 1, 2$, for solutions (Φ^1, Φ^1) and $\left(\Phi^1, \Phi^{-1}\right)$ is made in Fig. 5. One can see that the change of partial energies $F_1(l)$ of long layers in all range of l does not exceed several percent. On the other hand, the graphs of partial energies of short layers $F_2(l)$ practically coincide till l_B. Hence, the full energy of $\left(\Phi^1, \Phi^{-1}\right)$ bound state is less than the full energy of $\left(\Phi^1, \Phi^1\right)$ — the couple (Φ^1, Φ^{-1}) is more stable than (Φ^1, Φ^1). This means that in the experiment the probability of detection of $\left(\Phi^1, \Phi^1\right)$-state is less than the probability of detection of $\left(\Phi^1, \Phi^{-1}\right)$-state.

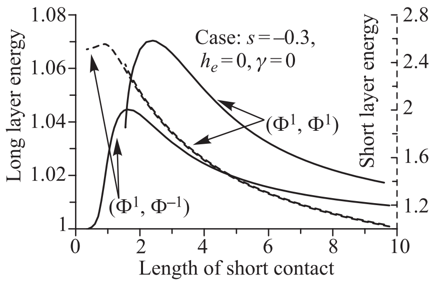

Fig. 5. Comparison of the partial energies for (Φ^1, Φ^1)- and (Φ^1, Φ^{-1})-states

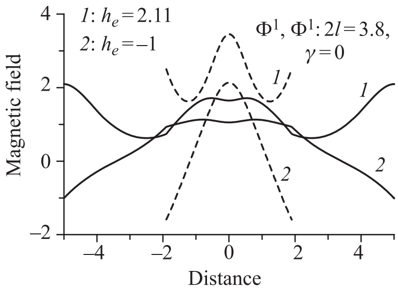

Fig. 6. Graphs of (Φ^1, Φ^1) in critical values of h_e

In addition, we shall note that the values of average magnetic fluxes (3.1) are $N\left[\Phi^{\pm 1}\right] = \pm 1$ in all the admissible range of l, so the derivatives $\partial N_i(l)/\partial l = 0$, $i = 1, 2$.

Every magnetic flux distribution $\varphi(x)$ in JJ has a region of existence by change of h_e for fixed values of other parameters. This region is limited by lower $h_{\min}$ and upper $h_{\max}$ values of the external magnetic field. If $\gamma = 0$ the values $h_{\min}$ and $h_{\max}$ are called critical fields for solution under consideration [6]. For unipolar solutions the points $(h_{\min}, 0)$ and $(h_{\max}, 0)$ on the plane (h_e, γ) are bifurcation points at change of h_e, where the transitions from Josephson to resistive regimes take place.

In Fig. 6 the distributions of magnetic fields for (Φ^1, Φ^1)-type solution of (1.4) for $2l = 3.8$, $h_{\max} \approx 2.11$ (curves *1*) and $h_{\min} \approx -1$ (curves *2*) are demonstrated. It can be seen that the external magnetic field h_e changes the long layer distribution $\varphi_{1,x}(x)$ mainly in the neighborhood of the boundaries $\pm L$. At the same time, the deformation of the internal magnetic field $\varphi_{2,x}(x)$ in the short layer is considerable along the whole length.

CONCLUSIONS

A spline-collocation scheme for numerical solution of non-linear BVP for systems of ODEs given on the embedded intervals is worked out. The scheme realization leads to a system of the block-diagonal system of algebraic equations. Such a scheme can be easily extended to problems with discontinuous coefficients without violation of the structure of algebraic system.

The developed technique gives possibilities for detailed investigation of many multiparametric physical problems. Especially, we analyze the existence and stability of some types of magnetic flux distributions in magnetically coupled 2-layer JJs.

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