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PHYSICAL GENERAL RELATIVITY

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Физическая теория относительности

Мы формулируем принцип общей ковариантности и восстанавливаем в общековариантной форме фундаментальные понятия электрического и магнитного полей, выводим общековариантные уравнения Максвелла для этих полей и узнаем, что общековариантное фундаментальное понятие интервала определяется не группой Лоренца, а общековариантной зеркальной симметрией. Это определяет прямой путь к объяснению природы времени и пространства и выражает сущность физической теории относительности. Из геодезических уравнений мы смогли вывести общековариантные уравнения Ньютона и, следовательно, восстановить базисные понятия скорости и импульса, силы, работы и энергии в общековариантной форме. Этим мы раскрыли связи физической теории относительности с классической механикой, решив при этом известную проблему нулевого гамильтониана.

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Physical General Relativity

We formulate the Principle of General Covariance and restore the fundamental notions of electric and magnetic fields in the general covariant form, derive the general covariant Maxwell equations for these fields and recognize that the general covariant fundamental notion of interval is determined not by the Lorentz group, but by a general covariant bilateral symmetry. This directly leads from electromagnetism to a new understanding of the nature of time and space and expresses the essence of physical general relativity. Considering the geodesic equations, we are able to derive the general covariant Newton equations and, hence, to restore the basic notions of velocity and momentum, force, work, and energy in the general covariant form. Thus, we disclose relations of the physical general relativity with classical mechanics and give a natural solution to the known problem of the zero Hamiltonian.

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INTRODUCTION

Our goal here is to formulate the Principle of General Covariance, restoring in the general covariant form the fundamental notions of electric and magnetic fields and the Maxwell equations of these fields in order to lay the fundamentals of physical general relativity. In the framework of physical general relativity, we also exhibit that the so-called geodesic equations of general relativity theory contain, in the hidden general covariant form, the Newton equations and the concepts of physical velocity, momentum, force, work, and energy. This gives an adequate solution to the known zero-Hamiltonian problem and ensures, in the evident form, the transition to the special theory of relativity.

The paper is organized as follows. In Sec. 1, all aspects of the Principle of General Covariance are presented on the basis of the seminal papers [1] and [2]. Our goal is to obtain a clear and exhaustive definition of this fundamental principle of nature and not just of the general theory of relativity. There is too intimate connection between gravity and the rest to be considered separately. That is why, the Principle of General Covariance takes on fundamental significance as a top idea. In what follows, the words "general covariant" will mean "defined in the framework of the Principle of General Covariance".

In Sec. 2, we consider the general covariant generalization of the concept of interval in the context of the problem of the general covariant definition of electric and magnetic fields and give the solution of the problem in question. Section 3 is devoted to the derivation of the general covariant Maxwell equations of electric and magnetic fields and definitions of natural time and physical space in the framework of the Principle of General Covariance. On this ground, in Sec. 4, we derive the general covariant Newton equations and restore the physical notions of physical velocity, momentum, force, work, and energy in the general covariant form. In Conclusions, we pay attention to the heuristic physical aspects of our consideration.

1. PRINCIPLE OF GENERAL COVARIANCE

In this section, the Principle of General Covariance is formulated as a fundamental principle of nature itself with the emphasis on the work by Einstein and Grossmann published in 1913 [1] and on Einstein's subsequent work [2], where the general relativity (reparameterization symmetry) was actually discovered. The first original paper contains the first systematic representation in the physical literature of the adequate mathematical formalism proposed by Ricci and Levi-Civita.

The Principle of General Covariance includes three aspects: the definition of general covariant fields that can only be used to formulate the most general laws and principles of symmetry of nature; the definition of general relativity (reparameterization symmetry); the definition of natural derivative. “General covariant” everywhere means “approved by the Principle of General Covariance”.

The definition of general covariant fields presupposes some acquaintance with the theory of functions of many variables, an important element of which is the reference space R^n, which provides a reliable geometrical basis and a definite geometrical visibility. Since the field of real numbers R is continuous and unconditional to all forms of physical matter, we can define on this ground the continuous and unconditional natural geometry R^n, in which a point is defined as an n-tuple of real numbers $x = (x^1, x^2, \cdots, x^n)$, $\quad -\infty < x^i < \infty$, and the distance function is introduced as usual $d(x,y) = \sqrt{(x^1 - y^1)^2 + (x^2 - y^2)^2 \cdots + (x^n - y^n)^2}$. If $a \in R$ and $b \in R$, then $a + b \in R$ and, hence, the points $x = (x^1, x^2, \cdots, x^n)$ and $y = (y^1, y^2, \cdots, y^n)$ are not distinguished, since it is always possible to write $y^i = x^i + a^i$. It is clear that R^1, R^2, R^3 can be considered as numerical models of the Euclidian straight line, plane, and space, respectively. However, R^1, R^2, R^3 admit a simple and clear generalization and, hence, R^n is a very important geometry, which can be considered as the underlying structure of any investigation. The geometry unfolded in the reference space R^n gives a universal and general method for constructing geometries with nontrivial topology in which a point is defined as that of some n-dimensional surface in the space R^N, $n < N$. These generalized geometries can be put into correspondence with definite states of the general covariant physical system. The details of this correspondence will be considered in a suitable place.

The numbers x^i are called the Cartesian coordinates of the point x. This is the initial system of coordinates. It is clear that the Cartesian coordinates $x^1, x^2, \cdots, x^n$ of the point R^n should be considered on the absolutely equal footing from any point of view. Hence, it is impossible to introduce the so-called space coordinates and the time coordinate in the framework of the reference space R^n alone. But instead of this, we can look for a definition of natural time as an entity that is tightly connected with the hidden essence of gravity and electromagnetism. This is the main goal and motivation of our consideration.

Functions $f(x^1, x^2, \cdots, x^n)$ can be defined not only in all the reference space but just in its part Ω called a domain without boundary. The set Ω is defined as follows. If a point $x = (x^1, x^2, \cdots, x^n)$ belongs to Ω and there exists $\varepsilon > 0$ such that for a point $y = (y^1, y^2, \cdots, y^n)$ the inequalities

$$|y^i - x^i| < \varepsilon$$

are fulfilled, then y belongs to Ω. A domain with boundary is Ω with limit points which form its boundary. Each function f has the domain of its definition and the range of its values and is continuously differentiable.

If the variables $\overline{x}^1$, $\overline{x}^2$, $\cdots, \overline{x}^n$ are not the Cartesian coordinates, then they are some auxiliary variables (called curvilinear coordinates) connected with the Cartesian one x^1, $x^2, \cdots, x^n$ by the nonlinear relations

$$x^i = x^i(\overline{x}^1, \cdots, \overline{x}^n), \quad \overline{x}^i = \overline{x}^i(x^1, \cdots, x^n), \quad \frac{\partial x^i}{\partial \overline{x}^k}\frac{\partial \overline{x}^k}{\partial x^j} = \delta^i_j,$$

which are given here in the implicit form.

Now it is time to consider the first aspect of the Principle of General Covariance. Let a system of m functions $V^1(x), \cdots, V^m(x)$ of independent variables x^1, $\cdots, x^n$ be given. The rule is defined (the law of transformation) which shows how one can uniquely find another system of m functions $\overline{V}^1(\overline{x}), \cdots, \overline{V}^n(\overline{x})$ of curvilinear coordinates $\overline{x}^1$, $\cdots, \overline{x}^n$ only through the functions of the first system and the derivatives of $\overline{x}^i$ with respect to x^j $(\partial \overline{x}^i / \partial x^j)$ and (or) the derivatives of x^j with respect to $\overline{x}^i$ $(\partial x^k / \partial \overline{x}^j)$, which can be called the functions of transformation. It is clear that these functions are primary entities in the framework of the Principle of General Covariance.

If $m = n$ and the law of transformation has the form

$$\overline{V}^i(\overline{x}) = V^j(x)\frac{\partial \overline{x}^i}{\partial x^j}, \tag{1}$$

then one says that the contravariant tensor field of type $(1, 0)$ is defined. The term "tensor field" can be explained as follows. In physics, the term "field" is tied to the physical reality that expands all over our space or its region. Thereby, a field is in general described by the functions of three independent variables. In our case, a tensor field is expanded all over some region of the reference space R^n, which is the region of definition of the considered system of functions. Hence, the relevance of the use of the word "field" is possible in all cases in question. The necessity to consider the region is also dictated by derivatives that define the law of transformation (1). The so-defined tensor field is also characterized by the general covariant autonomous system of equations that determines the congruence of lines in the reference space and the partial differential equation

$$\frac{dx^i}{dt} = V^i(x), \quad V^i(x)\frac{\partial F(x)}{\partial x^i} = 0.$$

The last equation is general covariant if $F(x)$ is a tensor field of type $(0, 0)$, that is the general covariant field which transforms as follows:

$$\overline{F}(\overline{x}) = F(x). \tag{2}$$

A tensor field of type $(0, 0)$ is usually called a scalar field. Scalar fields have an interesting peculiarity: they can be constant since the needed condition of constancy is $\partial_j F(x) = 0$ and it does not depend on the choice of an arbitrary system of coordinates. Indeed, in accordance with (2), for the gradient of a scalar field, the law of transformation has the following form:

$$\frac{\partial \overline{F}(\overline{x})}{\partial \overline{x}^j} = \frac{\partial F(x)}{\partial x^k}\frac{\partial x^k}{\partial \overline{x}^j}$$

that defines the covariant tensor field of type $(0,1)$. The tensor field $A_i(x)$ of type $(0,1)$ until the law of transformation

$$\overline{A}_j(\overline{x}) = A_k(x)\frac{\partial x^k}{\partial \overline{x}^j} \tag{3}$$

is also characterized by the integral $\int A_i dx^i$ along the path in the reference space and the general covariant equation $\partial_i A_j(x) - \partial_j A_i(x) = 0$, which expresses in the general covariant form the condition of independence of the integral from the choice of the path between two given points.

The tensor fields of type $(1,0)$ are generally reffered to as vector fields for the following reason. The general covariant structure of the linear space naturally appears on the set of the scalar and vector fields. Indeed, let $a(x)$, $b(x)$ be two scalar fields and $U^i(x)$, $V^i(x)$ be two vector fields, then $W^i(x) = a(x)U^i(x) + b(x)V^i(x)$ is evidently again the vector field. This general covariant linear space has notable properties. For a natural and constructive introduction of the general covariant linear operator, we have no need to introduce n linear independent vector fields $E^i_\mu(x)$ (a frame field) as in the abstract theory of linear spaces, since the role of the general covariant linear operator is played here by the tensor field $S^i_j(x)$ of type $(1,1)$. We consider the general covariant equation $\overline{V}^i(x) = S^i_j(x)V^j(x)$ as the definition of the linear operator in the linear space in question. It is also very important to recognize that this equation defines the fundamental representation of the general covariant gauge group of internal symmetry with the associative product $P^i_j(x) = S^i_k(x)T^k_j(x)$. The considered general covariant gauge group essentially uniquely defines the nature of gravity and, hence, the right-hand side of the Einstein equation. The details can be found in [3].

The general covariant field $P^i_{jk}(x)$ with the law of transformation of the form

$$\overline{P}^i_{jk}(\overline{x}) = P^l_{mn}(x)\frac{\partial \overline{x}^i}{\partial x^l}\frac{\partial x^m}{\partial \overline{x}^j}\frac{\partial x^n}{\partial \overline{x}^k} + \frac{\partial \overline{x}^i}{\partial x^l}\frac{\partial^2 x^l}{\partial \overline{x}^j \partial \overline{x}^k} \tag{4}$$

is called a connection. The connection is not transformed as a tensor. Nevertheless, the transformed components depend only upon the components before the transformation and transformation functions.

A general conclusion means that only general covariant fields considered as invariant systems of functions, defined and classified by arbitrary transformations of coordinates, should be used to formulate the most general and fundamental laws of nature. Such general covariant fields are called tensor fields and connections. The definition of a tensor field of type (p,q) is an evident generalization of formulas (1) and (3). We must always remember about the dual nature of general covariant fields, which is reflected in their definition. On the one hand, these quantities (similar to the electromagnetic field) are invariant and, hence, their properties are coordinate-free and basis-free. The same general covariant quantity can be presented in any system of coordinates. On the other hand, the general covariant quantities are systems of functions of many variables, and this is very important, since the fundamental laws of nature are formulated as systems of differential equations in partial derivatives. Let us consider a simple analogy to illustrate our message. A number system is similar to a system of coordinates. A transition from one number system to another is analogous to a transformation of coordinates. However, the properties of numbers are formulated in the invariant form (independently of the definite choice of a number system). On the contrary, the definition of general covariant quantities include transformations of coordinates, but all properties of these quantities are independent of the choice of a coordinate system. However, such a permutation does not change the essence of the matter. We can expect that the interplay between an observer and computing machines which is provided by the binary number system has an analog in the framework of the Principle of General Covariance (a preferred system of coordinates in nature itself). We emphasize once again that a function is a primary entity of the Principle of General Covariance.

Einstein discovered the general relativity (reparameterization symmetry) in the work [2], even before the equations of the gravitational field were established in the final form. To define the reparameterization symmetry as one more aspect of the Principle of General Covariance, let us consider a domain Ω in the reference space R^n. By the transformation of this domain we will mean $2n$ real functions $\alpha^i(x^1, \cdots, x^n) = \alpha^i(x), \quad \alpha^i_{-1}(x^1, \cdots, x^n) = \alpha^i_{-1}(x)$ for which the domains of their definition and the ranges of their values coincide with Ω, and moreover, $\alpha(\alpha_{-1}(x)) = x$, $\alpha_{-1}(\alpha(x)) = x$. Thus, under the transformation of Ω we put each of its points x into correspondence with the point y so as to $y^i = \alpha^i(x), \quad \alpha^i_{-1}(y) = x^i$.

Let us consider a new system of coordinates $\overline{x}^i = \overline{x}^i(x) = \alpha^i(x)$ in the domain Ω which is defined by the transformation of this domain. Let $A_i(x)$ be a covector field in Ω. In the new system of coordinates, one can consider the covector field $A_i(\overline{x})$. Let us see how this new covector field looks like in the initial system of coordinates. We have

$$\widetilde{A}_i(x) = A_j(\overline{x})\frac{\partial \overline{x}^j(x)}{\partial x^i} = A_j(\alpha(x))\frac{\partial \alpha^j(x)}{\partial x^i}.$$

Thus, we put any transformation of the domain Ω into correspondence with the transformation of the covector fields in accordance with the rule

$$\widetilde{A}_i(x) = A_j(\alpha(x))\frac{\partial \alpha^j(x)}{\partial x^i}.$$

For the scalar field and the symmetric covariant tensor field of the second rank, we have $\widetilde{\varphi}(x) = \varphi(\alpha(x))$,

$$\widetilde{g}_{ij}(x) = g_{kl}(\alpha(x)) \frac{\partial \alpha^k(x)}{\partial x^i} \frac{\partial \alpha^l(x)}{\partial x^j}.$$

Einstein uncovered that if $g_{ij}(x)$ is the solution of the general covariant equations, then $\widetilde{g}_{ij}(x)$ will be the solution of the same equations as well. This means that the standard parameterization of the solutions through the initial data is unfit. Such an unusual situation dramatically turned the investigation of the true equations of the gravitational field on a false track.

It is not difficult to write the laws of reparameterization for any general covariant fields. We mention only two important aspects of the reparameterization symmetry. The variation of the covector field under the reparameterization can be represented as difference $\triangle A_i(x) = \widetilde{A}_i(x) - A_i(x)$. Let us consider some infinitesimal transformation of the domain Ω, setting $\alpha^i(x) = x^i + \xi^i(x)$ with condition that $\xi^i(x)$ should be trivial on the boundary of Ω. One can prove that $\xi^i(x)$ is the vector field. It is not difficult to show that under the infinitesimal transformation $\triangle A_i(x) = \widetilde{A}_i(x) - A_i(x) = \xi^k(x)\partial_k A_i(x) + A_k(x)\partial_i \xi^k(x)$. The right-hand side of this equation is called the Lie derivative. It should be emphasized that in the general covariant theory it is natural to consider the Lie derivative as a variation in the stationary-action principle.

General relativity (reparameterization symmetry) means that in the general covariant theory the true physical configuration corresponds not to a unique set of fields, but to a whole class of reparameterization equivalent configurations.

The concept of natural derivative is the last but not least aspect of the Principle of General Covariance which is mainly motivated by the demand that covariant derivatives should not enter into the canonical Lagrangians of the physical fields [4]. The well-known example of the canonical Lagrangian represents the Lagrangian of the electromagnetic field.

Let us give the general definition of natural derivative and with this the selection rule that provides a possibility to find a very restricted number of the general covariant fields in their infinite manifold.

If some general covariant field is given and from the components of this field and its partial derivatives one can form (using the algebraic operation only) a new general covariant field, then the connection between these fields is called the natural derivative.

The gradient of the scalar field is the simplest but important example. The tensor of the electromagnetic field is the natural derivative of the covector field.

The notion of natural derivative has an exceptional meaning since it emphasizes that the symmetric tensor is the unique general covariant field. Indeed, if we take the components of the symmetric covariant tensor field g_{ij} and form its derivatives $\partial_j g_{kl}$, then these derivatives are not the components of a general covariant field. However, from g_{ij} and these partial derivatives one can form (with the help of algebraic operations only) a new general covariant field (the Christoffel symbols)

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} (\partial_j g_{kl} + \partial_j g_{kl} - \partial_l g_{jk}) \tag{5}$$

with the law of transformation given by (4). We see that the Christoffel symbols are a natural derivative of g_{ij} which defines the Levi-Civita connection. The Riemann tensor is a natural derivative of this connection

$$R_{ijl}{}^k = \partial_i \Gamma^k_{jl} - \partial_j \Gamma^k_{il} + \Gamma^k_{im} \Gamma^m_{jl} - \Gamma^k_{jm} \Gamma^m_{il}. \tag{6}$$

Thus, the symmetric Ricci tensor $R_{jl} = R_{kjl}{}^k$ can be considered as the second-rank natural derivative of g_{ij}. The unique status of the gravitational field and the Einstein law of gravity $R_{jl} = 0$ is demonstrated. There is no other general covariant field with such a property.

Resume. The Principle of General Covariance includes the definition and classification of the kinematically possible general covariant fields in which arbitrary transformations of the coordinates play the key role, as well as the definition of general relativity (reparameterization symmetry) and the selection rule of unique general covariant fields by the method of natural derivative.

The Principle of General Covariance does not contain assertions about the physical content of the general covariant laws of nature, but speaks about necessary and sufficient elements of its mathematical formulation which includes the principle of general covariant symmetries, the important example of which was exhibited above.

2. MOTIVATION FOR OUR STUDY

The general covariant electric and magnetic fields disappeared from the consideration in the general theory of relativity where we deal with the general covariant tensor of the electromagnetic field $F_{ij} = \partial_i A_j - \partial_j A_i$. Such “integration” of electric and magnetic fields cannot be considered as satisfactory, since condensators and magnets evidently exist, and this fact should be put into correspondence with the Principle of General Covariance as the fundamental principle of nature itself. The tensor of the electromagnetic field has six independent components that is equal to the number of independent components of electric and magnetic fields. The problem is to find one-to-one general covariant correspondence between these quantities. As the first evident step in the needed direction, we can simply put for the electric field $E_i = t^k F_{ik}$. But we know nothing about the nature and physical sense of the vector field t^i.

To clarify this entangled situation, we can pay attention to the fact that general relativity in a certain sense is a far-reaching generalization of the special theory of relativity. That is why we can start with the question: “What is the general covariant generalization of the concept of interval of the special theory of relativity?” The answer is given by the formula

$$ds^2 = \overline{g}_{ij} dx^i dx^j = (\frac{2t_i t_j}{g_{kl} t^k t^l} - g_{ij}) dx^i dx^j,$$

which can be found, for example, in [5]. From this formula it follows that

$$\overline{g}_{ij} = \frac{2t_i t_j}{g_{kl} t^k t^l} - g_{ij}, \quad t_i = g_{ij} t^j. \tag{7}$$

We see that in the general case, the Einstein potential $\overline{g}_{ij}$ is the function of a vector field t^i and a symmetric tensor field g_{ij}, which defines the natural general covariant scalar product in the general covariant linear space of vector fields

$$(t|u) = g_{ij} t^i u^j, \quad (t|t) = g_{ij} t^i t^j \geqslant 0, \quad (t|t) = 0, \quad \text{if and only if} \;\; t^i = 0.$$

Thus, for the scalar field

$$\varphi = \frac{(t|u)}{\sqrt{(t|t)}\,\sqrt{(u|u)}},$$

we have $-1 \leqslant \varphi \leqslant 1$, but it is evident that in general we cannot speak about the angle between two vector fields, since from the equation $\cos\alpha = \varphi$ it follows that α is a function of x. We can only prove that there are orthogonal vector fields. Indeed, if $(t|u) \neq 0$, we put $v^i = u^i - (t|u)t^i/(t|t)$ and, hence, t^i and v^i are orthogonal $(t|v) = 0$ for any x.

Now our goal is to ensure the internal content of Eq. (7). The right-hand side of Eq. (7) contains fourteen unknown functions. It is clear that this number should be reduced to ten unknown functions of the Einstein potential $\overline{g}_{ij}$. The first natural step is to put $t_i = \partial_i f, \quad t^i = g^{ij} t_j$. To simplify relation (7), we can write the equation

$$g^{ij} \partial_i f \partial_j f = 1, \tag{8}$$

which can be considered as a general covariant generalization of the equation of geometrical optics. Other arguments in favour of this equation will be presented in what follows. Constraint (8) provides in its final form the formal solution of the problem in question, and for the Einstein potential we have

$$\overline{g}_{ij} = 2\partial_i f \partial_j f - g_{ij}, \quad t^i = g^{ij} \partial_j f, \quad g^{ij} \partial_i f \partial_j f = 1. \tag{9}$$

We will call $f(x)$ a scalar constituent of the Einstein potential and g_{ij} — the Riemann constituent.

Now we will give the geometrical interpretation of the positive definite quadratic differential form (the Riemann constituent) $dl^2 = g_{ij} dx^i dx^j$. Let us consider the four-dimensional surface $X^a = F^a(x^1, x^2, x^3, x^4) = F^a(x)$ in the reference space R^N with the Cartesian coordinates X^a. If the functions $g_{ij}(x)$ are known, we can solve the system of equations

$$\delta_{ab}\frac{\partial F^a(x)}{\partial x^i}\frac{\partial F^b(x)}{\partial x^j}=g_{ij}(x)$$

and find the unknown functions $F^a(x)$ and dimension N. Thus, we can put into correspondence with any solution of the Einstein equation a curved surface on which the length of any curves is defined by the Riemann functional $\int dl=\int\sqrt{g_{ij}dx^i dx^j}$. By this, we provided a natural realization of Einstein's idea about gravity as a curved space.

And, last but not least, we need to explain a hidden (and, hence, very important) reason of appearance of the indefinite quadratic differential form $ds^2=\overline{g}_{ij}dx^i dx^j$. Let R^i_j be a linear operator and $(Ru)^i=R^i_k u^k$. Suppose that this operator is self-adjoint with respect to the natural scalar product $(u|v)$, $(Ru|v)=(u|Rv)$. From the last equation it follows that $g_{ik}R^k_j=g_{jk}R^k_i$ and, hence, the tensor field $g_{ik}R^k_j$ is symmetric. The idea is to connect the existence of the indefinite quadratic forms ds^2 with the self-adjoint operators defined as usual with respect to the natural (positive definite) scalar product. We demonstrate that this idea can be realized in our case and establish a connection between the symmetric tensor fields $g_{ik}R^k_j$ and $\overline{g}_{ij}$. We remind that at our disposal there are only the vector field t^i and the symmetric tensor field g_{ij}, which defines the natural scalar product in the general covariant linear space of the vector fields. We also pay attention to that there is a fundamental discrete symmetry of nature known as bilateral symmetry which is defined by the given vector and the well-known scalar product in familiar vector algebra.

On the ground of this association, we give the natural general covariant definition of the bilateral symmetry in the framework of the Principle of General Covariance as follows. A pair of vector fields v and $\overline{v}$ has general covariant bilateral symmetry with respect to the gradient of the scalar constituent of the Einstein potential t^i, if the sum of these fields is collinear to t^i and their difference is orthogonal to it, $\overline{v}+v=$ $=\lambda t,\quad (\overline{v}|t)=(v|t)$, where $(u|v)$ is a scalar product defined above and, hence, $\overline{v}^i=(2t^i t_j-\delta^i_j)v^j,\quad v^i=(2t^i t_j-\delta^i_j)\overline{v}^j$. From this definition it follows that the connection between the right-hand sided and left-hand sided vector fields can be represented as a linear transformation (reflection) $\overline{v}^i=R^i_j v^j$, $R^i_j=2t^i t_j-\delta^i_j,\quad R^i_k R^k_j=\delta^i_j,\quad \mathrm{Det}(R^i_j)=-1$. The vector t^i and the tensor g_{ij} are invariant under reflection since $R^i_j t^j=t^i,\quad R^i_k R^j_l g_{ij}=g_{kl}$.

Now it is easy to see that the needed connection is given by the equations $\overline{g}_{ij}=g_{ik}R^k_j=2t_i t_j-g_{ij},\quad \overline{g}^{ij}=g^{ik}R^j_k=2t^i t^j-g^{ij}$. These fundamental equations are a strong argument to put forward an idea that the general covariant bilateral symmetry is a strict and fundamental symmetry of nature and, hence, in all natural physical processes, a right-hand sided general covariant physical quantity (let it be v) always appears in pairs with the left-hand sided one $\overline{v}$.

Now our initial motivation looks like as heuristic since we recognized that in general the fundamental concept of interval is clearly defined by the general covariant bilateral symmetry and not the Lorentz group. Moreover, now it is clear how to introduce the concepts of electric and magnetic fields in the framework of the Principle of General Covariance.

3. GENERAL COVARIANT ELECTRIC AND MAGNETIC FIELDS

Let e_{ijkl} be the antisymmetric Levi-Civita tensor associated with g_{ij}, $e_{1234} = \sqrt{g}$, where $g = \mathrm{Det}\,(g_{ij}) > 0$. For the contravariant antisymmetric tensor $e^{ijkl} = g^{im}g^{jn}g^{kr}g^{ls}e_{mnrs}$, $e^{1234} = \sqrt{1/g}$. For the tensor of the electromagnetic field F_{ij}, we put $\widetilde{F}_{ij} = \frac{1}{2}e_{ijkl}F^{kl}$, $F^{kl} = g^{ki}g^{lj}F_{ij}$. General covariant electric and magnetic fields are introduced as follows:

$$E_i = t^k F_{ik}, \quad H_i = t^k \widetilde{F}_{ik}. \tag{10}$$

It is evident from the definition that $(t|E) = (t|H) = 0$. Equation (10) can be inverted. Let us prove that

$$F_{ij} = -t_i E_j + t_j E_i - e_{ijkl}t^k H^l. \tag{11}$$

We have

$$-e_{ijkl}t^k H^l = -e_{ijkl}t^k t_m \widetilde{F}^{lm} = -e_{ijkl}e^{lmrs}\frac{1}{2}t^k t_m F_{rs}.$$

Since $-e_{ijkl}e^{lmrs} = \delta^{mrs}_{ijk}$, where δ^{mrs}_{ijk} is the Kronecker symbol, then

$$-e_{ijkl}t^k H^l = t^k(t_i F_{jk} + t_j F_{ki} + t_k F_{ij}) = t_i E_j - t_j E_i + F_{ij},$$

as $t^i t_i = 1$ and this is an additional argument to introduce the constraint (8).

Now we are ready to derive the Maxwell equations for the general covariant electric and magnetic fields from the equations for the tensor of the electromagnetic field F_{ij}, which are considered in the general theory of relativity. Beforehand, it is important to formulate basic relations of the natural generalization of familiar vector algebra and vector calculus in the framework of the Principle of General Covariance.

The scalar product $(A|B)$ of the vector fields A^i and B^i was defined above. The vector product $C = [A \times B]$ is defined as follows:

$$C^i = [A \times B]^i = e^{ijkl}t_j A_k B_l, \quad A_k = g_{kl}A^l.$$

We suppose that g_{ij} and t_i are known. It is evident that $[A \times B] + [B \times A] = 0$. The main relations of general covariant vector algebra

$$|[A \times B]| = |A||B|\sin\alpha, \quad [A \times [B \times C]] = B(A, C) - C(A, B)$$

are fulfilled. We also state that

$$[ABC] = [BCA] = [CAB], \quad [ABC] = (A|[B \times C]).$$

The basic operators of the general covariant vector calculus are defined quite naturally

$$\operatorname{div} A = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\,A^i), \quad (\operatorname{grad}\phi)^i = g^{ij}\partial_j\phi,$$
$$\operatorname{div}\operatorname{grad}\phi = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\,g^{ij}\partial_j\phi) = \nabla_i\nabla^i\phi,$$

where ∇_i is a covariant derivative in the Levi-Civita connection (5) belonging to g_{ij}.

The rotor of the vector field A is introduced as a vector product of the four-dimensional operator ∇ and A

$$(\operatorname{rot} A)^i = [\nabla \times A]^i = e^{ijkl}t_j\partial_k A_l = \frac{1}{2}e^{ijkl}t_j(\partial_k A_l - \partial_l A_k).$$

We have

$$\operatorname{rot}\operatorname{grad}\phi = 0, \quad \operatorname{div}\operatorname{rot} A = 0.$$

Let $\overline{\Gamma}^i_{kj}$ be the Levi-Civita connection of the Einstein potential $\overline{g}_{ij}$. From (5), (8), and (9) we find that $\overline{\Gamma}^i_{kj} = \Gamma^i_{kj} + 2t^i\nabla_k t_j$. Hence, it follows that in the general theory of relativity, the equations for the tensor of the electromagnetic field can be written in the following form:

$$\nabla_i\widetilde{F}^{ij} = 0, \quad \nabla_i\overline{F}^{ij} = 0, \tag{12}$$

where $\widetilde{F}^{ij} = \frac{1}{2}e^{ijkl}F_{kl}, \quad \overline{F}^{ij} = \overline{g}^{ik}\overline{g}^{jl}F_{kl}$.

We have

$$\widetilde{F}^{ij} = -t^iH^j + t^jH^i - e^{ijkl}t_kE_l, \quad \overline{F}^{ij} = t^iE^j - t^jE^i - e^{ijkl}t_kH_l. \tag{13}$$

Substituting (13) into (12), we derive the general covariant Maxwell equations for electric and magnetic fields

$$\operatorname{rot} E = -\frac{1}{\sqrt{g}}D_t(\sqrt{g}\,H), \quad \operatorname{rot} H = \frac{1}{\sqrt{g}}D_t(\sqrt{g}\,E), \tag{14}$$
$$\operatorname{div} E = 0, \quad \operatorname{div} H = 0, \tag{15}$$
$$(t|E) = 0, \quad (t|H) = 0, \tag{16}$$

where D_t is the operator of the Lie derivative defined above $(D_tE)^i = t^k\partial_kE^i - E^k\partial_kt^i$.

Let us reproduce the derivation of the first and fourth equations. From (12) and (13) we have

$$-\nabla_i\widetilde{F}^{ij} = \nabla_i(t^iH^j - t^jH^i + e^{ijkl}t_kE_l) =$$
$$= t^i\nabla_iH^j - H^i\nabla_it^j + H^j\nabla_it^i - t^j\nabla_iH^i + e^{jkil}t_k\nabla_iE_l.$$

Since

$$(D_t H)^i = t^k \partial_k H^i - H^k \partial_k t^i = t^k \nabla_k H^i - H^k \nabla_k t^i,$$
$$D_t \sqrt{g} = \partial_k(\sqrt{g}\, t^k) = \sqrt{g}\, \nabla_k t^k,$$

then

$$-\nabla_i \widetilde{F}^{ij} = \frac{1}{\sqrt{g}} D_t(\sqrt{g}\, H)^j + e^{jkil} t_k \nabla_i E_l - t^j \nabla_i H^i.$$

Taking into account that $D_t t_i = D_t t^i = 0$ and, hence, $t_j D_t(\sqrt{g}\, H)^j = 0$, we derive from the last formula what we need.

From equations (14) we can derive

$$D_t(\partial_i(\sqrt{g}\, H^i)) = 0, \quad D_t(\partial_i(\sqrt{g}\, E^i)) = 0,$$
$$D_t(\sqrt{g}\, t_i E^i) = 0, \quad D_t(\sqrt{g}\, t_i H^i) = 0,$$

and therefore Eqs. (14)–(16) are compartible.

For the energy–momentum tensor of the electromagnetic field, we have the following representations:

$$T_{ij} = \frac{1}{2} g_{ij}(E^2 + H^2) - E_i E_j - H_i H_j - t_i \Pi_j - t_j \Pi_i,$$
$$E^2 = (E|E), \quad H^2 = (H|H), \tag{17}$$

where $\Pi_i = g_{ij}\Pi^j$ — the components of the general covariant Poynting vector

$$\Pi^i = e^{ijkl} t_j E_k H_l, \quad \Pi = [E \times H].$$

For the energy density, we have

$$\varepsilon_m = t^i t^j T_{ij} = \frac{1}{2}(E^2 + H^2).$$

If $D_t g_{ij} = 0$, from the Maxwell equations (14)–(16) we can derive the law of energy conservation

$$\frac{1}{\sqrt{g}} D_t(\sqrt{g}\, \varepsilon_m) + \nabla_i \Pi^i = 0.$$

Resume. The fundamental physical notions of electric and magnetic fields are put into correspondence with the Principle of General Covariance to have a deeper perception of nature. There is no doubt that the general covariant Maxwell theory of electric and magnetic fields opens the door to the unexplored world of physical general relativity. It is clear from our consideration that the scalar constituent $f(x)$ of the Einstein potential (9) has an exact relation to the enigma of natural time (time of nature itself). To make it easier to perceive the definitions given below, let us appeal to physical intuition. We know very well the physical phenomena connected with the temperature and pressure difference. We speak about the gradient of temperature and pressure and presuppose that values of these physical quantities are known for any point of some region of the Euclidian space.

From a geometrical point of view, we deal here with a scalar field that is invariant with respect to all admissible transformations of coordinates. Now it is reasonable to suppose that there are a field of moments of natural time and an area of phenomena defined by the gradient of time. However, following the principle of sufficient reason, we consider a simple but important realization of our general consideration, and after that exact definitions will be given.

We consider the four-dimensional reference space R^4 with the metric $dl^2 = g_{ij}dx^i dx^j = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2, \quad g_{ij} = \delta_{ij}$ and look for solutions to Eq. (8) under this condition. Equation (8) reads as

$$\left(\frac{\partial f}{\partial x^1}\right)^2 + \left(\frac{\partial f}{\partial x^2}\right)^2 + \left(\frac{\partial f}{\partial x^3}\right)^2 + \left(\frac{\partial f}{\partial x^4}\right)^2 = 1 \tag{18}$$

and in accordance with our general statement has a general solution $f(x) = a_i x^i = a_1 x^1 + a_2 x^2 + a_3 x^3 + a_4 x^4$, where $a_i = t_i$ and, hence, $(a|a) = 1$, and a special solution $f(x) = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2}$. For the Einstein potential (9), we have in the first case $\overline{g}_{ij} = 2a_i a_j - \delta_{ij}$ and $ds^2 = (2a_i a_j - \delta_{ij})dx^i dx^j$. We can put into correspondence this quadratic differential form, the quadratic form $s^2 = (2a_i a_j - \delta_{ij})x^i x^j$ in R^4, which defines the three-parameter family of the Lorentz groups. The case of the Lorentz group $a_i = (0, 0\,0, 1)$ is investigated very well. We believe that now it is time to formulate adequate and key definitions of physical general relativity.

Definition: a moment of natural time is a number that we put into correspondence with any point of the reference space R^n. Hence, a moment of time is defined by the equation $t = f(x^1, x^2, \cdots, x^n) = f(x)$, and $f(x)$ is identified with the scalar constituent of the Einstein potential (9). It is very important to emphasize at this point that a temporal scalar field is invariant with respect to general coordinate transformations.

By definition, all points of the reference space that correspond to the same moment of time constitute physical space $S(t)$. A point of $S(t)$ is defined by the equation $f(x^1, x^2, \cdots, x^n) = f(x) = t = \text{constant}$ and, hence, a change of states of physical systems is connected with variation of values of the function $f(x)$ that is put into correspondence with natural time. Actually, this was demonstrated by the example of the Maxwell equations for electric and magnetic fields.

The gradient of time is the vector field t with the components $t^i = (\nabla f)^i = g^{ij}\partial_j f = g^{ij}t_j$, which defines fundamental discrete internal symmetry — general covariant bilateral symmetry defined above. The Einstein potential $\overline{g}_{ij} = 2t_i t_j - g_{ij} = g_{ik}R^k_j, \quad \overline{g}^{ij} = 2t^i t^j - g^{ij}$ provides a straightforward method of considering dynamical processes through the introduction of natural time into the Lagrangians (and the equations) of the fundamental physical fields. The idea is put forward that bilateral symmetry is a strict symmetry of nature and, hence, in all physical processes one cannot distinguish the right-hand sided physical quantity from the left-hand sided one.

From the consideration of the bilateral symmetry it follows that in the general covariant form, the time-reversal invariance means that a theory is invariant with respect to the transformation T: $t^i \to -t^i$. The transformation T has meaning if and only if the domains of values of the potentials $f(x)$ and $-f(x)$ coincide. In accordance with this definition, a theory will be time-reversal invariant if the gradient of the temporal field appears in all formulae only as an even number of times, like $t^i t^j$.

Since the temporal field enters into the Lagrangians of the physical fields in the form of the gradient of the scalar field $t_i = \partial_i f(u)$, the laws of nature are invariant with respect to transformations of the form $f(x) \Rightarrow f(x) + a$, where a is a constant. This symmetry defines the law of energy conservation as a fundamental physical law of nature itself, which is true in all cases.

The potential $f(x)$ of natural time is a solution to Eq. (8), which can be considered as a definition of uniformity of natural time. Other mathematical arguments in favour of this equation are also impressible. Let $dx^i = t^i dt$, $f(x + dx) = t + dt$, $f(x) = t$, then $df(x) = t_i t^i dt = dt$ and, hence, $t_i t^i = 1$. This is Eq. (8). Further, we consider the differential operator $D_t = t^i \partial_i$ defined by the gradient of natural time t^i and its exponent $\exp(aD_t) = 1 + aD_t + \frac{a^2}{2}(D_t)^2 + \cdots$. We put forward a natural demand that the transformation $f(x) \Rightarrow f(x) + a$ is generated by the exponent of the gradient of natural time t^i, and from equation $\exp(aD_t) f(x) = f(x) + a$ we again derive Eq. (8). Taking into account the possibility of changing the scale, we also subordinate the potential $f(x)$ of natural time to the equation

$$f(\lambda x^1, \lambda x^2, \cdots, \lambda x^n) = \lambda f(x^1, x^2, \cdots, x^n). \tag{19}$$

The fundamental (from a physical point of view) observation reads that Eq. (8) has not only a general solution but also a special solution known as the function of geodesic distance. This means that there are two different times in nature and, hence, two different kinds of natural dynamical processes. To illustrate this important general statement, let us go back to Eq. (18) and its general and special solutions. From the equations $f(x) = a_i x^i = a_1 x^1 + a_2 x^2 + a_3 x^3 + a_4 x^4 = t = \text{constant}$ and $f(x) = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2} = \tau = \text{constant}$ we see that, in one case, physical space is the familiar three-dimensional Euclidian space E^3 and, in the other, a new physical space is the three-dimensional sphere S^3. We can see that there are two different times in the same reference space R^4. The physical (mass) points are to be identified with the points belonging to the three-dimensional Euclidian space E^3, but the points belonging to the three-dimensional sphere S^3 should be put into correspondence with the Spherical Tops. Indeed, the symmetries of the Euclidian space can be composed of translations and rotations, and the symmetries of the three-dimensional sphere S^3 coincide with those of the Spherical Top. In other words, geometrical points in the Euclidian and spherical spaces have different physical meanings. The concept of the Spherical Top can be reduced to the concept of the point particle but in the dual time and dual physical space. A natural rotation is a motion in dual time. Thus, from the duality of time it follows that any known particle can be put into correspondence with a dual particle (dparticle) moving in the dual time. Hence, it is natural to put forward the idea of dual approach to the world of elementary particles which can explain the existence of leptons and quarks, lepton–quark symmetry and confinement (if we identify dparticles with quarks).

4. FROM GEODESIC EQUATIONS TO THE NEWTON EQUATIONS

The general covariant notions of velocity, momentum, force, work, and energy are not used in the general theory of relativity where the only so-called geodesic equations are considered. Such a situation is not satisfactory from a physical point of view since we know that the gravitational field performs work and this fact should be put into correspondence with the Principle of General Covariance to expand the boundaries of physical general relativity outlined above. Thus, our main task here is to restore the just listed fundamental physical concepts in the framework of the Principle of General Covariance. To this end, we consider the motion of a massive charged point particle in the external gravitational and electromagnetic fields via the general covariant action principle. We derive the general covariant and reparameterization invariant Newton equations from the Euler–Lagrange equations which represent the so-called geodesic equations. Path reparameterization is defined as a one-to-one and smooth transformation of the definition region of the parameter along a path in the reference space. On the basis of the natural concept of time, general covariant definitions of physical velocity, momentum, force, work, and energy (invariant with respect to the path reparameterization) are established. The connection between the change of the orientation of a path and the operation of the charge conjugation is marked. For the first time, one can see the results and visible changes that the Principle of General Covariance and the path reparameterization carry into the classical mechanics, and thereby expand the boundaries of physical general relativity.

The general covariant operator of the rate of change of the gravitational field with time is the Lie derivative along the gradient of time

$$D_t g_{ij} = t^l \partial_l g_{ij} + g_{lj} \partial_i t^l + g_{il} \partial_j t^l = \nabla_i t_j + \nabla_j t_i = 2\nabla_i t_j, \qquad (20)$$

where ∇_i is the covariant derivative with respect to the Levi-Civita connection belonging to g_{ij}. The gravitational field is called static if its rate of change with time is trivial $D_t g_{ij} = 2\nabla_i t_j = 0$. It is clear that these definitions are general covariant.

Let us consider the class of functions of one variable $\tau = \alpha(\sigma)$, for which the region of their definition coincides with the region of their values. If σ is running $[a, b]$, then $\alpha(\sigma)$ does this as well. Each function of this class defines the same path in the reference space R^n in accordance with the equations $x^i = x^i(\alpha(\sigma)), i = 1, 2, \cdots, n$. For a graphic representation of a set of functions $\tau = \alpha(\sigma)$, one should use the plane σ, τ with fixed points (a, a), (a, b), (b, b), (b, a). The decreasing functions connect the points (a, b), (b, a) and the increasing ones connect the points (a, a), (b, b). The simplest graphics are defined by the functions $\tau = \sigma$, $\tau = b - (\sigma - a)$. For the functions $\alpha(\sigma)$ in question, the following conditions are fulfilled: $d\alpha/d\sigma > 0$, $d\alpha/d\sigma < 0$, which define the orientation of the path. In the first case, $\alpha(a) = a, \quad \alpha(b) = b$, whereas $\alpha(a) = b$, $\alpha(b) = a$ in the case of the path with the opposite orientation. The equations of motion of a point particle are invariant with respect to the path reparameterization if the functions $x^i = x^i(\alpha(\sigma)), i = 1, 2, \cdots, n$, are the solutions of these equations under any $\alpha(\sigma)$.

Let us consider the autonomous system of equations $dx^i/d\sigma = v^i(x^1, \cdots, x^n)$ and suppose that $x^i = x^i(\sigma)$, $i = 1, 2, \cdots, n$, is its solution. Since

$$\frac{dx^i(\alpha(\sigma))}{d\sigma} = \frac{dx^i(\alpha(\sigma))}{d\alpha(\sigma)}\frac{d\alpha(\sigma)}{d\sigma} = v^i(x^1(\alpha(\sigma)), \cdots, x^n(\alpha(\sigma))\frac{d\alpha(\sigma)}{d\sigma},$$

then $x^i = x^i(\alpha(\sigma))$, $i = 1, 2, \cdots, n$, is again a solution but only under the condition that $d\alpha(\sigma)/d\sigma = 1$. Hence, the system of equations in question is invariant only with respect to the transformations $\sigma \to \sigma + a$, where a is constant. It is not invariant with respect to the path reparameterization. It is clear that the Hamilton equations are not invariant with respect to the path reparameterization, and that is why it is very important, from a physical point of view, to consider in all the details the transition from the Euler–Lagrange equations, which are general covariant and invariant with respect to the path reparameterization, to the Hamilton equations in the case of the motion of a massive charged particle in the external gravitational and electromagnetic fields.

We investigate the Lagrangian

$$L = -m\sqrt{\overline{g}_{ij}u^i u^j} - \alpha A_i u^i = L_1 + L_2, \tag{21}$$

where $u^i = dx^i/d\sigma$, m denotes the mass of a point particle to which we put in correspondence the path $x^i = x^i(\alpha(\sigma))$, $i = 1, 2, \cdots, n$. The coordinates and parameter σ have the dimension of length. The action is dimensionless and, hence, the mass m and components of the electromagnetic potential have the dimension of inverse length, and the constant of interaction α is dimensionless. We clarify this choice of dimensions by the equation $(e/c)A_i = \hbar(e^2/\hbar c)(1/e)A_i$ from which it follows that $(1/e)A_i$ has the dimension of the inverse length. Thus, under this choice of dimensions, the operation of charge conjugation looks like

$$\mathrm{C} : A_i \to \mathrm{C}A_i = -A_i. \tag{22}$$

We investigate the invariance of the action with respect to the path reparameterization. Since $\sigma \subset [a, b]$ and $x^i = x^i(\sigma)$, we write the action in the form of definite integral

$$S = \int_a^b L(\sigma)d\sigma = \int_a^b L_1(\sigma)d\sigma + \int_a^b L_2(\sigma)d\sigma = F_1(b) - F_1(a) + F_2(b) - F_2(a),$$

where $F_1(\sigma)$ and $F_2(\sigma)$ are the primitives of $L_1(\sigma)$ and $L_2(\sigma)$, respectively. For $x^i = x^i(\alpha(\sigma))$, we write the action in the following form:

$$\widetilde{S} = \int\limits_a^b \widetilde{L}(\sigma)d\sigma.$$

Since $\widetilde{L}_1(\sigma) = L_1(\alpha(\sigma))|d\alpha(\sigma)/d\sigma|$, $\widetilde{L}_2(\sigma) = L_2(\alpha(\sigma))d\alpha(\sigma)/d\sigma$, we will distinguish two cases. If $d\alpha/d\sigma > 0$, then $\alpha(a) = a$, $\alpha(b) = b$ and, hence, $\widetilde{S} = F_1(\alpha(b)) - F_1(\alpha(a)) + F_2(\alpha(b)) - F_2(\alpha(a)) = S$. For $d\alpha/d\sigma < 0$, $\alpha(a) = b$, $\alpha(b) = a$ and, hence, $\widetilde{S} = -F_1(\alpha(b)) + F_1(\alpha(a)) + F_2(\alpha(b)) - F_2(\alpha(a)) = S$ only under the condition that the path reparameterization is accompanied by the charge conjugation (22). We can see that the change of the path orientation is tightly connected with the charge conjugation.

The Euler–Lagrange equations $\delta S = 0$ can be written as follows:

$$\frac{d}{d\sigma}\left(-m\frac{\overline{g}_{ij}u^j}{\sqrt{\langle u|u\rangle}} - \alpha A_i\right) = -\frac{m}{2}\frac{\partial\overline{g}_{jk}}{\partial x^i}\frac{u^j}{\sqrt{\langle u|u\rangle}}u^k - \alpha\frac{\partial A_k}{\partial x^i}u^k,$$

where $\langle u|u\rangle = \overline{g}_{ij}u^i u^j$. After some transformations, these equations can be written in the following form:

$$\frac{dQ^i}{d\sigma} + \overline{\Gamma}^i_{kj}u^k Q^j = \alpha\,\overline{F}^i_k u^k, \tag{23}$$

where

$$Q^i = \frac{mu^i}{\sqrt{\langle u|u\rangle}}, \quad \overline{F}^i_k = \overline{g}^{il}F_{lk},$$

and $\overline{\Gamma}^i_{kj}$ are the Christoffel symbols of the Einstein potential $\overline{g}_{ij}$. If Γ^i_{kj} are the Christoffel symbols of g_{ij}, then $\overline{\Gamma}^i_{kj} = \Gamma^i_{kj} + 2t^i\nabla_k t_j$. The last relation and concept of natural time is the starting point to derive from Eqs. (23) the general covariant Newton equations (Newton's second law) invariant with respect to the path reparameterization $x^i = x^i(\sigma) \to x^i = x^i(\alpha(\sigma))$.

We define that some quantity (which is given along the path $x^i = x^i(\sigma)$) is invariant with respect to the path reparameterization if it does not depend on the factor $d\alpha(\sigma)/d\sigma$. It is evident that all physical fields are invariant with respect to the path reparameterization if the path goes through the region of definition of these fields. It is easy to see that the vector Q^i is invariant with respect to the path reparameterization, but u^i is not. It is necessary to strictly distinguish the path reparameterization and the introduction of a new parameter $\overline{\sigma} = \varphi(\sigma)$.

The path reparameterization changes the orientation of the path if $d\alpha/d\sigma < 0$. Let $x^i = x^i(\sigma)$ be a solution of Eqs. (23), then $x^i = x^i(\alpha(\sigma))$ is again a solution of the same equations if $d\alpha/d\sigma > 0$. In the opposite case, $d\alpha/d\sigma < 0$

and the functions $x^i = x^i(\alpha(\sigma))$ will be a solution of Eqs. (23) only after the charge conjugation (22). Indeed, for $x^i = x^i(\sigma)$, we have

$$Q^i(\sigma) = \frac{m\,dx^i(\sigma)}{\sqrt{\overline{g}_{ij}(x(\sigma))dx^i(\sigma)dx^j(\sigma)}}$$

and, hence, for $x^i = x^i(\alpha(\sigma))$, we obtain

$$\widetilde{Q}^i(\sigma) = \frac{m\,dx^i(\alpha(\sigma))}{\sqrt{\overline{g}_{ij}(x(\alpha(\sigma))dx^i(\alpha(\sigma))dx^j(\alpha(\sigma))}} = Q^i(\alpha(\sigma))\varepsilon,$$

$$\varepsilon = \frac{d\alpha(\sigma)/d\sigma}{|d\alpha(\sigma)/d\sigma|} = \pm 1.$$

We can see a direct connection between the charge conjugation and the orientation of a path of a massive charged particle. The symmetry of this kind attracts attention since there is an evident but unclear discrepancy between matter and antimatter in the Universe.

Now we are strongly motivated to consider the decomposition

$$Q^i = Wt^i + \pi^i, \quad (t|\pi) = t_i\pi^i = 0, \tag{24}$$

and derive equations for W and π^i from Eqs. (23). We have

$$\frac{dQ^i}{d\sigma} + \overline{\Gamma}^i_{kj}u^kQ^j = \frac{d\pi^i}{d\sigma} + \overline{\Gamma}^i_{kj}\pi^ju^k + W(\frac{dt^i}{d\sigma} + \overline{\Gamma}^i_{kj}t^ju^k) + \frac{dW}{d\sigma}t^i.$$

Taking into account the relations $t^k\nabla_kt^i = t^k\nabla_it^k = 0$, $Wu^k = (t_lu^l)Q^k$, it is easy to check that

$$W(\frac{dt^i}{d\sigma} + \overline{\Gamma}^i_{kj}t^ju^k) = (t_lu^l)\pi^k\nabla_kt^i. \tag{25}$$

We put

$$\frac{d\pi^i}{d\sigma} + \overline{\Gamma}^i_{kj}\pi^ju^k = \frac{d\pi^i}{d\sigma} + \widetilde{\Gamma}^i_{kj}\pi^ju^k + (\pi^j\nabla_jt_ku^k)t^i = \frac{D\pi^i}{d\sigma} + (\pi^j\nabla_jt_ku^k)t^i$$

and have $t_iD\pi^i/d\sigma = 0$, where $\widetilde{\Gamma}^i_{kj} = \Gamma^i_{kj} + t^i\nabla_kt_j$. We conclude that $D\pi^i/d\sigma$ can be considered as a rate of the change of the physical momentum with respect to the parameter σ.

To complete our investigation, we transform the right-hand side of Eqs. (23) as well. The general covariant definition of electric and magnetic fields is given by the relations (10). With (11) we derive

$$\overline{F}^i_ku^k = -t^iE_ku^k - (t_ku^k)E^i - [u\times H]^i, \quad [u\times H]_i = e_{ijkl}t^ju^kH^l.$$

We conclude that Eqs. (23) can be written as a system of two equations

$$\frac{dW}{d\sigma} + \pi^i\nabla_it_ku^k + \alpha E_ku^k = 0, \tag{26}$$

$$\frac{D\pi^i}{d\sigma} + (t_l u^l)\pi^j \nabla_j t^i + \alpha(t_l u^l)E^i + \alpha[u \times H]^i = 0. \tag{27}$$

We see that there is a small asymmetry in the expression for the Lorentz force F^i due to the factor $t_i u^i$ and this is a very important argument to introduce the fundamental concept of physical velocity. We write the sequence

$$u^i = (t_l u^l)\frac{u^i}{t_l u^l} = (t_l u^l)(\frac{u^i}{t_l u^l} - t^i + t^i) = (t_l u^l)(v^i + t^i)$$

and have a beautiful expression for the Lorentz force

$$F^i = (t_l u^l)\Bigl(\alpha E^i + \alpha[v \times H]^i\Bigr)$$

since $[t \times H] = 0$. Thus, the definition of physical velocity with respect to the parameter σ is given by the expression

$$v^i = \frac{u^i}{t_l u^l} - t^i, \quad t_i v^i = 0, \tag{28}$$

which is general covariant and invariant with respect to the path reparameterization. Other evidences in support of this definition of the physical velocity can be presented as follows. Since $u^i = (t_l u^l)(v^i + t^i)$, then $\langle u|u\rangle = |t_l u^l|\sqrt{1-v^2}$ and, hence,

$$Q^i = \varepsilon\biggl(\frac{mv^i}{\sqrt{1-v^2}} + \frac{mt^i}{\sqrt{1-v^2}}\biggr), \quad \varepsilon = \frac{|t_l u^l|}{t_l u^l} = \pm 1.$$

We conclude that

$$\pi^i = \frac{\varepsilon m v^i}{\sqrt{1-v^2}}, \quad W = \frac{\varepsilon m}{\sqrt{1-v^2}}. \tag{29}$$

At last, we have the following expression for the Lagrangian (21):

$$L = (t_l u^l)\Bigl(-\varepsilon m\sqrt{1-v^2} - \alpha\Phi_i v^i - e\varphi\Bigr),$$

where $\varphi = (t|A) = t^l A_l$ is the scalar potential of the electromagnetic field and $\Phi_i = A_i - t_i\varphi$ is its vector potential $(t|\Phi) = t^i\Phi_i = 0$ expressed in the general covariant form.

Equations (26) and (27) read as

$$\frac{1}{t_l u^l}\frac{dW}{d\sigma} + \pi^i \nabla_i t_k v^k + \alpha E_k v^k = 0, \tag{30}$$

$$\frac{1}{t_l u^l}\frac{D\pi^i}{d\sigma} + \pi^k \nabla_k t^i + \alpha E^i + \alpha[v \times H]^i = 0. \tag{31}$$

The factor $1/t_l u^l$ provides the invariance of the Newton general covariant equations (30) and (31) with respect to the path reparameterization. Equation (30) expresses the law of energy conservation and Eq. (31) represents the Newton second law defined by the Principle of General Covariance.

From Eqs. (30) and (31) we have the following expression for the gravitational force:

$$F_g^i = \pi^k \nabla_k t^i = \pi^k P_k^i, \tag{32}$$

which is general covariant (if F_g^i is equal to zero in some system of coordinates, then it will be trivial in any system of coordinates). We conclude that the gravitational force is not trivial only for the nonstatic gravitational fields. Remind that for the static gravitational field, $D_t g_{ij} = 2\nabla_i t_j = 0$, where ∇_i is the covariant derivative with respect to the connection belonging to g_{ij}. This is an unknown and unexpected result. We believe that it can be verified easily.

We have obtained the general covariant and invariant with respect to the path reparameterization expressions for the physical velocity, momentum, force, work, and energy of massive charged particles moving in the external gravitational, electric and magnetic fields.

Now we pay attention to the fact that transformation, which defines the physical velocity through the component of tangential vector u^i, is not reversible, because v^i has three independent components and u^i has four independent components. It is evident that we can put $t_l u^l = 1$ and the transition from the Euler–Lagrange equations to the Hamilton equations will be trivial. However, we need to estimate the content and consequences of this condition. We see that the equation $t_l u^l = 1$ is general covariant but not invariant with respect to the path reparameterization. A reason is as follows. For the natural time, we have $t = f(x^1, x^2, x^3, x^4)$. Hence, along the path $x^i = x^i(\sigma)$, $\quad t = f(x^1(\sigma), x^2(\sigma), x^3(\sigma), x^4(\sigma))$ and $dt = t_i u^i d\sigma = d\sigma$. Thus, we see that under the condition that $t_l u^l = 1$, the path is automatically parameterized by the moments of natural time. We can speak about the trajectory of motion in this case, which is defined by the equations

$$\frac{dW}{dt} + \pi^i \nabla_i t_k v^k + \alpha E_k v^k = 0, \tag{33}$$

$$\frac{D\pi^i}{dt} + \pi^k \nabla_k t^i + \alpha E^i + \alpha [v \times H]^i = 0, \tag{34}$$

where

$$\frac{D\pi^i}{dt} = \frac{d\pi^i}{dt} + \widetilde{\Gamma}^i_{kj} \pi^j \frac{dx^k}{dt}$$

is the general covariant definition of the rate of the physical momentum change with respect to natural time. For the physical momentum and energy, we have the same equations (29) but with the following definition of the physical velocity v^i with respect to natural time:

$$v^i = \frac{dx^i}{dt} - t^i \tag{35}$$

and $\varepsilon = 1$,

$$\pi^i = \frac{m v^i}{\sqrt{1 - v^2}}, \quad W = \frac{m}{\sqrt{1 - v^2}}. \tag{36}$$

Thus, the concept of natural time at the fundamental level provides a full correspondence of physical general relativity with the classical mechanics and special relativity and uncovers its hidden so far deep physical content.

If $v^i = 0$, then $dx^i/dt = t^i$. This system of equations defines the congruence of curves called the lines of time. When a particle moves along a line of time, its "shadow" on the surface of physical space is at rest. Hence, any movement has invisible and visible components. We can only make assumptions about the nature of invisible component.

CONCLUSIONS

Thus, it is shown that physical general relativity uncovers the deep physical content of the Einstein gravity theory with the new concept of natural time. Here it announces itself by the Principle of General Covariance, the general covariant Maxwell equations (14)–(16) for electric and magnetic fields, and the general covariant Newton equations (30) and (31). It restores the fundamental physical meaning of electric and magnetic fields, velocity, momentum, force, work, and energy in the framework of the Principle of General Covariance. It unravels the puzzles of time and predicts the duality of natural time. It should be emphasized here that the duality of time demonstrates that in a certain sense the well-known idea of "rotating rigid body" (also mentioned as the Top) of classical mechanics is as fundamental as the idea of "mass point", i. e., the first concept can be reduced to the second one at the fundamental (field-theoretical) level and this opens terra incognita. The prediction of the general covariant gravitational force (32), which is not trivial only for the nonstatic gravitational fields, may have practical meaning in the vicinity of the Sun. We would also like to emphasize the connection between the path orientation and charge conjugation. The adequate solution to the well-known problem of zero Hamiltonian should be mentioned as a visible achievement of physical general relativity.

We believe that physical general relativity provides a needed unified basis for the physical world as a whole.

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