Finiteness of generalized Chern−Simons thoeries

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We study the perturbation theory for the example of a topological Batalin–Vilkovisky theory and show that it is free from the UV divergences. In fact, the property of finiteness can be generalized for some class of theories, which we describe here. 3–dimensional Chern–Simons and 2–dimensional topological Yang–Mills theories belong to this class.

Key words: BV formalism, topological theory, Chern–Simons, BF

1 Introduction

At the last decade it became obvious, that BV formalism and homological algebra can be useful tools for investigating a big amount of theories. Among them there are SUSY YM and CS [5]. In the current work we describe a simple model of the BV theory, which is topological and in particular case coincides with the CS theory.

This text is organized as follows. First we formulate concerned theory, which we call ω–theory. After a brief overview of the BV formalism we show the way of quantizing of ω–thoery, build the perturbation theory and prove that this theory is free from the UV divergences. At the end we give a slight generalization of obtained results.

2 ω−theory

The action of ω–theory is:

$$S = \mathrm{Tr}\int_{M_d} \omega\,\mathrm{d}\omega + \frac{2}{3}\,\omega^3\,. \tag{1}$$

Here we used the following notations:

M_D is a compact D–dimensional manifold without boundary, $D = 2N + 1$, $N \in \mathbb{N}$.

$\omega \in \Omega^\bullet(M_d, g)$ — i.e ω is a polyform with values in g (which is a Lie algebra for this theory). More explicitly:

$$\begin{aligned}\omega = \omega^{(0)} + \omega^{(1)} + \ldots + \omega^{(d)} = \omega^{(0)} + \mathrm{d}x^{i_1}\omega^{(1)}_{i_1} + \frac{1}{2!}\,\mathrm{d}x^{i_1}\,\mathrm{d}x^{i_2}\omega^{(2)}_{i_1 i_2} + \\ + \cdots + \frac{1}{d!}\,\mathrm{d}x^{i_1}\ldots\mathrm{d}x^{i_d}\omega^{(d)}_{i_1\ldots i_d}\,.\end{aligned} \tag{2}$$

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It is useful to introduce the following $\mathbb{Z}_2$–gradings:

$$\mathrm{gh}(\omega^{(k)}_{i_1\dots i_k}) = 0, \text{ if } \omega^{(k)}_{i_1\dots i_k} \text{ is boson field;}$$
$$\mathrm{gh}(\omega^{(k)}_{i_1\dots i_k}) = 1, \text{ if } \omega^{(k)}_{i_1\dots i_k} \text{ is fermion field;}$$

$$\deg(\omega^{(k)}) = k \bmod 2;$$

$$\epsilon\big(\omega^{(k)}\big) = \big(\mathrm{gh}(\omega^{(k)}) + \deg(\omega^{(k)})\big) \bmod 2\big).$$

We impose an additional condition on ω: $\epsilon(\omega) = 1$.

3 Quantization

We will quantize this theory using the BV–formalism [1], [3].

Let's briefly remind the procedure of BV–quantizing.

– Consider some set of boson and fermion fields Φ^A, $A = \overline{1, N}$ and the same set of fields Φ^*_A, but with opposite statistics: $\mathrm{gh}(\Phi^A) = \big(\mathrm{gh}(\Phi^*_A) + 1\big) \bmod 2$ (here gh is just an indicator of fermion fields and such notation is used for coinciding with other notations in this paper)

– BV–action is a function of this fields ($S = S(\Phi, \Phi^*)$) and satisfies BV–equation:
$$\frac{1}{2}\,(S,S)_{BV} = \mathrm{i}\hbar\Delta_{BV}S \;\Leftrightarrow\; \Delta_{BV}\mathrm{e}^{\mathrm{i}S/\hbar} = 0\,. \tag{3}$$
Here
$$(F,G)_{BV} = F\left(\frac{\overleftarrow{\partial}}{\partial\Phi^A}\frac{\overrightarrow{\partial}}{\partial\Phi^*_A} - \frac{\overleftarrow{\partial}}{\partial\Phi^*_A}\frac{\overrightarrow{\partial}}{\partial\Phi^A}\right)G\,,$$
$$\Delta_{BV} = \frac{\overrightarrow{\partial}}{\partial\Phi^*_A}F\frac{\overleftarrow{\partial}}{\partial\Phi^A}\,.$$
In general case BV–bracket $(\cdot,\cdot)_{BV}$ and BV–Laplacian Δ_{BV} are constructed by using the symplectic structure Ω. Here $\Omega = \delta\Phi^A \wedge \delta\Phi^*_A$.

– Using such action, we can build the quantum field theory. For this in the path integral we integrate not over all fields, but only over a Lagrangian submanifold: $Z = \int_{\mathcal{L}} \mathcal{D}\Phi\mathcal{D}\Phi^* e^{\mathrm{i}S/\hbar}$ (Lagrangian submanifold is a submanifold of maximal dimension, on which symplectic structure equal to zero). Such procedure is well-defined, and, in particular, variation over $\mathcal{L}$ equals zero.

Now we'll show that ω–theory is a BV–theory. Let's divide ω into two parts: $\omega = \omega_1 + \omega_2$ such, that: $\deg(\omega_1) = \mathrm{gh}(\omega_2) = 1$, $\deg(\omega_2) = \mathrm{gh}(\omega_1) = 0$.

Due to the Hodge duality ω_1 can be treated as Φ, and ω_2 as Φ^*.

The symplectic structure is $\Omega = \mathrm{Tr}\int \delta\omega_1 \wedge \delta\omega_2$.

Thus $(S,S)_{BV} \propto \mathrm{Tr}\displaystyle\int \frac{\delta S}{\delta\omega_1} \wedge \frac{\delta S}{\delta\omega_2}$, $\Delta_{BV} \propto \mathrm{Tr}\displaystyle\int \frac{\delta}{\delta\omega_1}\frac{\delta}{\delta\omega_2}S$.

A direct calculation yields: $\Delta_{BV}S = 0$ and $(S,S)_{BV} = 0$.

We define the Lagrangian submanifold by condition $\mathrm{d}^*\omega = 0$.

For further correct work we have to say some words about cohomology. Even after imposing the condition $\mathrm{d}^*\omega = 0$, operator d in the kinetic term, restricted to the Lagrangian submanifold, have a kernel. This kernel consists exactly of operator's cohomology. For taking this into account we will decompose ω into two parts: $\omega = \tilde{\omega} + \omega_0$, where ω_0 is a representative of the cohomology group. This decomposition is fixed uniquely by imposing an additional condition $\int \tilde{\omega}\omega_0 = 0$.

We will treat ω_0 as an external field (i.e. ω_0 becomes now a parameter of the theory). In particular, we will not integrate over it in the path integral.

Now we can define correlators in the quantized ω–theory:

$$\langle H \rangle = \int \mathcal{D}\tilde{\omega}\delta(d^*\tilde{\omega})H\mathrm{e}^{\mathrm{i}S(\tilde{\omega}+\omega_0)/\hbar}\,. \tag{4}$$

Note, that $\langle H \rangle$ will not depend on the Lagrangian submanifold, if $\Delta_{BV}\left(HE^{\mathrm{i}S/\hbar}\right) = 0$.

4 The perturbation theory

We'll build the perturbation theory treating interacting term as small. In fact, the expansion in this theory will go over $\hbar$. This becomes obvious after fields' rescaling $\omega \to \sqrt{\hbar}\,\omega$ and noticing, that parity of number of vertexes in all orders of perturbation theory is the same.

The perturbation theory is built absolutely in the standard way and we'll not describe it in details. This is not our aim. Vertexes here are obtained from $\mathrm{Tr}\int \omega^3$. So we have three vertexes: $\mathrm{Tr}\int \tilde{\omega}^3$, $\mathrm{Tr}\int \tilde{\omega}^2\omega_0$ and $\mathrm{Tr}\int \tilde{\omega}\omega_0^2$ (3–, 2–, and 1–valent respectively). After separation of cohomology and restriction on Lagrangian submanifold de Rahm operator became invertible. So the propagator in this theory is $(\mathrm{d}_{|\tilde{\omega}})^{-1}$ (for simplicity we will denote it further as d^{-1}).

Allocation of a Lagrangian manifold depends on metric. Thus d^{-1} depends on metric. In fact, $\mathrm{d}^{-1} = \dfrac{\mathrm{d}^*}{\Delta}_{|\tilde{\omega}}$.

Define a kernel of d^{-1} as

$$(\mathrm{d}^{-1}J)^a(x) = \int_{M_y} G^a_b(x,y) \wedge J^b(y)\,. \tag{5}$$

Here a, b are algebraic indices.

From the definition we see that $G \in \Omega^{d-1}(M \times M, g \otimes g^*)$.

For diagrams' calculation we will use point–splitting regularization. This means, that $G_\epsilon(x,y) = \begin{cases} 0\,, & \rho(x,y) \le \epsilon\,, \\ G(x,y)\,, & \rho(x,y) > \epsilon\,. \end{cases}$

5 3−dimensional Chern−Simons theory as a particular case of ω−theory

Let's consider the case, when $d = 3$. Puting $\omega_0 = 0$ and restricting on the Lagrangian submanifold, we can rewrite ω in the following form:

$$\tilde{\omega} = c + A + *\mathrm{d}b\,. \tag{6}$$

Here two fermion fields c and b (which are also a 0–forms) give us $\tilde{\omega}^{(0)}$ $(= c)$ and $\tilde{\omega}^{(1)}$ $(=*\mathrm{d}b)$. Boson 1–form A, which satisfies the condition $\mathrm{d}^*A = 0$, is just a $\tilde{\omega}^{(1)}$. $\mathrm{d}^*\tilde{\omega}^{(4)} = 0 \to \tilde{\omega}^{(4)} = 0$. So we can rewrite action in explicit form:

$$S(\omega) = \mathrm{Tr}\int_{M_3} A\,\mathrm{d}A + 2c\,\mathrm{d} * \mathrm{d}b + \frac{2}{3}\,A^3 + 2cA * \mathrm{d}b\,. \tag{7}$$

But this is just a BRST–fixed action of Chern–Simons theory in Lorentz gauge.

Thus this theories coincide up to the cohomologes.

It is well known, that Chern–Simons thoery is finite (see, for example [2]). Our main statement that ω–theory is also finite.

6 Proof of finiteness

Here we don't want to give a precise mathematical proof, but just give an idea, how this proof works. This proof is somehow a generalization of known one for the 3–dimensional case [2].

Near the diagonal of $M \times M$ (when x is close to y) we can introduce notations: $u^\mu \equiv x^\mu - y^\mu$, $\hat{u}^\mu \equiv u^\mu/||u||$.

It can be shown, that our propagator near the diagonal can be represented in the following way:

$$\begin{aligned} G^a_b(x,y) = \delta^a_b\epsilon_{\alpha_1\ldots\alpha_d}\big(&\hat{u}^{\alpha_1}\mathrm{d}\hat{u}^{\alpha_2}\ldots\mathrm{d}\hat{u}^{\alpha_d}\chi_0 + \hat{u}^{\alpha_1}\mathrm{d}\hat{u}^{\alpha_2}\ldots\mathrm{d}\hat{u}^{\alpha_{d-1}}\chi_1^{\alpha_d} + \\ &+\hat{u}^{\alpha_1}\mathrm{d}\hat{u}^{\alpha_2}\ldots\mathrm{d}\hat{u}^{\alpha_{d-2}}\chi_2^{\alpha_d-1\alpha_d} + \ldots + \hat{u}^{\alpha_1}\mathrm{d}\hat{u}^{\alpha 2}\ldots\chi_{d-2}^{\alpha_3\ldots\alpha_d}\big) + \lambda^a_b\,. \end{aligned} \tag{8}$$

Here $\chi_k^{\alpha_{d-k+1}\ldots\alpha_d}$ is a bounded k–form, λ^a_b is a bounded $(d-1)$–form.

Let's consider an arbitrary diagram with V vertexes. Vertexes are labelled with coordinates $x_1, \ldots, x_V$.

All divergences, which can arrive, come from the $\mathrm{d}\hat{u}$. $I_{\mathrm{div}}(\mathrm{d}\hat{u}) = 1$ (due to $\mathrm{d}\hat{u} \sim \dfrac{\mathrm{d}u}{u}$).

Each diagram can be represented as a sum of terms of the following form: $\int_{M^1}\ldots\int_{M^V}\mathrm{d}\hat{u}_{i_1}\ldots\mathrm{d}\hat{u}_{i_\sigma}\wedge$(something bounded). i_α is some index, which denote a kind of $\hat{u}$. The index of divergence of such term: $I_{\mathrm{div}}(\mathrm{term}) = \sigma - d(V-1)$. We are interested in the case, when it is bigger or equal to zero.

Let's introduce a notation $U = \mathrm{d}\hat{u}_{i_1}\ldots\mathrm{d}\hat{u}_{i_\sigma}$. From $\sigma \geq dV - d$ we obtain, that codimension of U is not bigger than d.

But we can find $d+1$ linearly independent vector fields, which are annihilated by U. This fields are

$$\sum_{i=1}^{V} \frac{\partial}{\partial x_i^\mu} \quad \text{and} \quad \sum_{i=1}^{V} x_i^\mu \frac{\partial}{\partial x_i^\mu}\,.$$

Thus $U = 0$.

So there is no terms which are potentially divergent.

7 Generalizations

In this prove we used the properties of the propagator near the diagonal and a specific form of vertexes, namely that this vertexes are a wedge product of polyforms. So we can generalize obtained result in the following way:

This theorem works for provisional theory with propagator d^{-1}:

- for arbitrary dimension (also for even d)
- for arbitrary vertex of the form $\mathrm{Tr}\int \phi \wedge \omega \ldots \wedge \omega$, where ϕ is an external parameter and ω is a polyform.

Another example of such theory is a BF–theory [4], action of which we can rewrite as:

$$S = \mathrm{Tr} \int_{M_d} \omega^* (\mathrm{d}\omega + \omega^2)\,. \tag{9}$$

Here $d \in \mathbb{N}$; $\omega^*, \omega \in \Omega^*(M_d, g)$; $\epsilon(\omega) = 1$,$\epsilon(\omega^*) = d \bmod 2$.

When $d = 2$, this theory is equivalent to the topological 2–dimensional Yang–Mills.

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